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Lecture Summaries for Multivariable Integral Calculus M52B

Lecture 01: Introduction. An introduction to the course, listing the topics to be covered; remarks concerning the types of proofs given throughout the course.

Lecture 02: Review of Areas under Curves and the Definite Integral. A review of the relationship between areas under curves and definite integration of a function of one variable; discussion of Riemann sum approximations to the area; definition of area and net signed area as a limit of Riemann sums; definition of the definite integral of a function; statement of theorems saying that continuity is a sufficient condition for integrability and that a definite integral can be computed by appropriate evaluation of an antiderivatives.

Lecture 03 (§16.1) : Volumes Under Surfaces and Double Integrals. Discussion of procedure for finding the volume of a solid lying between a region R in xy-plane and the graph of a non-negative function $f(x, y)$ defined on R; definition of the double integral of f over R, $\int\int_R f(x, y)\, dA$, in terms of a limit of Riemann sums, and its geometric interpretation as net signed volume; conditions on R and f which guarantee the existence of the double integral; theorem and proofs of the basic properties of double integrals.

Lecture 04 (§16.1) : Computing Double Integrals. Definition and examples of partial definite integration; definition of iterated integral; theorem and (geometric) proof that if $f(x, y)$ is continuous on the rectangular region $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy;$$

comments on the implicit assumptions about volume used in this geometric proof.

Lecture 05 (§16.2) : Double Integrals over Nonrectangular Regions: Theory. Discussion and examples of iterated integrals having the forms

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \qquad \text{and} \qquad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy;$$

definitions and examples of type I and type II regions; theorem and (geometric) proof that double integrals of continuous functions over these types of regions can be computed using these more general forms of iterated integrals.

Lecture 06 (§16.2) : Double Integrals over Nonrectangular Regions: Examples. Examples computing double integrals over non-rectangular regions of types I and II.

Lecture 07 (§16.3) : Polar Double Integrals: Theory. Definition, discussion, and examples of simple polar regions; definition of the polar double integral $\iint_R f(r, \theta)\, dA$ as a limit of polar Riemann sums; theorem and proof that if R is the simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the polar curves $r = r_1(\theta)$ and $r = r_2(\theta)$, and if $f(r, \theta)$ is continuous on R, then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta.$$

Lecture 08 (§16.3) : POLAR DOUBLE INTEGRALS: EXAMPLES. Examples computing polar double integrals.

Lecture 09 (§16.4) : PARAMETRIC REPRESENTATIONS OF SURFACES. Introduction to parametrization of surfaces; definitions of constant u-curves and constant v-curves; examples illustrating how to parametrize surfaces defined by equations where either x, y, or z is a function of the other two, or surfaces obtained by revolving the graph of $y = f(x)$ around the x-axis; examples illustrating how to find parametrizations with given constant u- or constant v-curves.

Lecture 10 (§16.4) : PARAMETRIC SURFACES: VECTOR EQUATIONS AND TANGENT PLANES. Discussion of the vector form of a parametrization for a surface; definition of continuity for vector-valued functions of two variables; definition of the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ of a vector-valued function $\mathbf{r}(u, v)$; geometric interpretation of these partials as tangent vectors to constant v- and constant u-curves respectively; derivation of the equation of the tangent plane to a parametric surface at a given point; technical remarks regarding existence and uniqueness of tangent planes; definition of the principal unit normal vector to a surface.

Lecture 11 (§16.4) : PARAMETRIC SURFACES: SURFACE AREA. Definition of smooth parametric surface; definition of the surface area of a smooth parametric surface; discussion of the motivation behind this definition.

Lecture 12 (§16.4) : EXAMPLES COMPUTING SURFACE AREA. Example showing that the surface area of a sphere of radius a is $4\pi a^2$; observation that if $f(x, y)$ is a differentiable function defined on domain D, then the surface area of the graph of f is given by $\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA$; example illustrating the observation.

Lecture 13 (§16.5) : TRIPLE INTEGRALS. Definition and discussion of the triple integral of a function of three variables over a solid region in 3-space; basic properties of triple integrals; theorem asserting that if G is the rectangular box defined by the inequalities $a \le x \le b$, $c \le y \le d$, $k \le z \le l$, and f is continuous on G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx,$$

and the iterated integral on the right can be replaced by any of the other 5 iterated integrals obtained by changing the order of integration; example illustrating the theorem.

Lecture 14 (§16.5) : TRIPLE INTEGRALS OVER MORE GENERAL REGIONS. Definition and example of a simple xy-solid; theorem and (sketch of) proof that if R is a region in the xy-plane and G is the simple xy-solid defined by $G = \{(x, y, z) \mid (x, y) \text{ in } R,\ g_1(x, y) \le z \le g_2(x, y)\}$ (where $g_1 \le g_2$ are continuous functions on R), then

$$\iiint_G f(x, y, z)\, dV = \iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz \right) dA,$$

assuming f is continuous on G.

Lecture 15 (§16.5) : TRIPLE INTEGRALS: EXAMPLES ON SIMPLE SOLIDS. Example illustrating the theorem proven in the previous lecture; corresponding definitions and theorems for simple xz-solids and simple yz-solids; example of triple integration over these types of solids.

Lecture 16 (§16.6) : MASS AND DENSITY OF A LAMINA. Definition and examples of lamina; definitions of homogeneous and inhomogeneous; definition of density for a homogeneous lamina; definition of density function for an inhomogeneous lamina; theorem and proof that if a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then the total mass M of the lamina is $\iint_R \delta(x, y)\, dA$; similarity between this result and the fundamental theorem of calculus; examples illustrating the theorem.

Lecture 17: (§16.6) CENTER OF GRAVITY. Discussions and derivations of the center of gravity for a system of two point-masses lying on a line, and a system of n point masses lying on a line.

Lecture 18: (§16.6) CENTER OF GRAVITY OF A LAMINA. Discussions and derivations of the center of gravity for a system of n point masses lying in a plane, and a lamina with mass distributed according to the density function $\delta(x, y)$.

Lecture 19: (§16.6) THEOREM OF PAPPUS AND CENTER OF GRAVITY OF A SOLID. Definition and discussion of the centroid of a lamina or region; proof of the theorem of Pappus relating the centroid of a region to the volume of a particular solid of revolution; definition of the density and density function for homogeneous and inhomogeneous solids in 3-space; theorem asserting that the total mass of a solid in 3-space is the triple integral of its density function; discussion of center of gravity for system of discrete particles in 3-space; formulas for the coordinates of the center of gravity of a solid in 3-space.

Lecture 20: (§16.7) TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES. Definition of cylindrical wedge; computation of the volume of a cylindrical wedge; definition of the triple integral in cylindrical coordinates of a function $f(r, \theta, z)$ over a solid G; statement of theorem expressing the triple integral in cylindrical coordinates as the appropriate triple iterated integral; discussion of the relationship between triple integrals in rectangular coordinates and triple integrals in cylindrical coordinates; example illustrating the theorem.

Lecture 21: (§16.7) TRIPLE INTEGRALS IN SPHERICAL COORDINATES. Definition of spherical wedge; statement of the volume of a spherical wedge; definition of the triple integral in spherical coordinates of a function $f(\rho, \theta, \phi)$ over a solid G; statement of theorem expressing the triple integral in spherical coordinates as the appropriate triple iterated integral; example illustrating the theorem; example illustrating the relationship between triple integrals in rectangular, cylindrical, and spherical coordinates.

Lecture 22: (§16.8) CHANGING VARIABLES IN INTEGRALS: PART I. Discussion and derivation of the change of variable formula for single integrals; definition of the Jacobian; statement of the change of variable formula for double integrals; definition, discussion, and example of a one-to-one transformation from the uv-plane to the xy-plane.

Lecture 23: (§16.8) CHANGING VARIABLES IN INTEGRALS: PART II. Definition of the Jacobian of a transformation; discussion of the relationship between the Jacobian of a transformation and the area of the image of a region in the domain; theorem and sketch of a proof of the change of variable formula for double integrals.

Lecture 24: (§16.8) CHANGING VARIABLES IN INTEGRALS: PART III. Examples illustrating how to apply the change of variables formula for double integrals, and why previous method for computing double integrals in polar coordinates is a special case of the change of variables formula; definitions of a transformation from uvw-space to xyz-space; definition of the Jacobian of such a transformation; statement of the change of variables theorem for triple integrals.

Lecture 25: (§17.1) VECTOR FIELDS. Definition of a vector field; examples of vector fields; discussion of graphical and coordinate representations of vector fields; definition of inverse-square fields.

Lecture 26: (§17.1) DEL, DIV, CURL, AND THE LAPLACIAN. Definitions, notations, and examples of the gradient (del), divergence, curl, and Laplacian operators; discussion of the coordinate independence of the divergence and the curl.

Lecture 27: (§17.2) LINE INTEGRALS WITH RESPECT TO ARC LENGTH. Definition and geometric interpretation of the line integral of f with respect to arc length (s) along a smooth curve C in 2-space; derivation of formula for evaluating a line integral with respect to arc length in the case where the curve C is parametrized by arc length; derivation of the more general formula for evaluating such a line integral directly from any smooth parametrization of the curve C; example illustrating this more general formula.

Lecture 28: (§17.2) OTHER TYPES OF LINE INTEGRALS. Definition of line integrals with respect to arc length in 3-space; formula for evaluating such line integrals; definition and discussion of line integrals with respect to x, y, and z; derivation of the formula for evaluating such integrals.

Lecture 29: (§17.2) PARAMETRIZATION INDEPENDENCE AND PIECEWISE SMOOTH CURVES. Proof that line integrals with respect to arc length along a curve do not depend on the parametrization or the orientation of that curve; proof that line integrals with respect to x, y and z do not depend on the parametrization of an *oriented* curve, but these line integrals change sign if the orientation of the curve is changed; definition of a piecewise smooth curve; definition of line integral along a piecewise smooth curve.

Lecture 30: (§17.2) LINE INTEGRALS AND WORK. Motivation for and definition, in terms of line integrals, of work done on a particle as it moves along a curve through a force (vector) field; remarks on why the work changes sign if the orientation of the curve is changed; derivation of formulas for evaluating the line integrals which compute work; discussion of various notations for these work integrals.

Lecture 31: (§17.3) CONSERVATIVE FIELDS AND INDEPENDENCE OF PATH: PART I. Statement and proof of the Fundamental Theorem of Work Integrals, implying that the value of a work integral of a conservative vector field along a piecewise smooth curve does not depend on the curve itself, but rather only on the endpoints of the curve; notation of the form $\int_{(x_0,y_0)}^{(x_1,y_1)} \mathbf{F} \bullet d\mathbf{r}$ to denote the work integral of the conservative field $\mathbf{F}$ along any piecewise smooth curve starting at (x_0, y_0) and ending at (x_1, y_1).

Lecture 32: (§17.3) CONSERVATIVE FIELDS AND INDEPENDENCE OF PATH: PART II. Definitions and examples of closed curve and connected set; theorem and proof that on a connected domain, $\mathbf{F}$ is conservative if and only if work integrals of $\mathbf{F}$ around piecewise smooth curves are zero if and only if work integrals of $\mathbf{F}$ are independent of path.

Lecture 33: (§17.3) CONSERVATIVE FIELD TEST. Definitions, examples, and discussions of simple curves and simply connected regions; statement and partial proof of the Conservative Field Test for 2-space; examples illustrating the theorem; example illustrating how to find a potential for a given conservative field; statement of the Conservative Field Test for 3-space; re-statement of this theorem in terms of curl.

Lecture 34: (§17.4) GREEN'S THEOREM FOR SIMPLY CONNECTED REGIONS. Statement and (partial) proof of Green's Theorem for simply connected regions; examples illustrating the usefulness of Green's Theorem; corollary showing that under appropriate conditions $f_y = g_x$ implies $f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative; corollary showing that $\text{Area}(R) = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C -y\,dx + x\,dy$, where C is a simple, closed, piecewise smooth curve, oriented counterclockwise, enclosing R; notation for line integrals around simple closed curves.

Lecture 35: (§17.4) GREEN'S THEOREM FOR MULTIPLY CONNECTED REGIONS. Definition of positive orientation; statement and proof of Green's Theorem for multiply connected regions; statement and proof of corollary that underlies the "Principle of Deformation of Path" in complex analysis.

Lecture 36: (§17.5) SURFACE INTEGRALS. Definition of surface integrals; derivation of formula for evaluating a surface integral over a smooth parametric surface; examples illustrating this formula and the relationship between surface area and surface integrals; derivation of formula for evaluating a surface integral where the surface is the graph of some function of x and y; example illustrating this formula.

Lecture 37: (§17.6) FLOWS AND ORIENTATIONS OF SURFACES. Informal definition of flux; discussion and examples of two-sided surfaces and one-sided surfaces (Mobius strip); definition of orientable and orientation; discussion of how a continuous unit normal vector field on a surface defines an orientation; examples of such fields for smoothly parametrized orientable surfaces; definitions of positive orientation, positive direction, negative orientation, and negative direction.

Lecture 38: (§17.6) FLUX. Derivation of the formula for flux of a vector field across an oriented surface; discussion of the evaluation of flux integrals in the cases where the surface is given parametrically and where the surface is the graph of some function of x and y.

Lecture 39: (§17.7) THE DIVERGENCE THEOREM. Definitions of closed surface and piecewise smooth surface; statement and partial proof of the divergence theorem.

Lecture 40: (§17.7) DIVERGENCE AND FLUX DENSITY. Example illustrating the Divergence Theorem; discussion of the physical interpretation of the divergence as the outward flux density (flux per unit volume) at a point; discussion of sources and sinks in an incompressible fluid; statement of Gauss's Law for Inverse-Square Fields.

Lecture 41: (§17.8) STOKES' THEOREM. Definition of positive orientation of a boundary curve relative to the orientation of the enclosed surface; statement of Stokes' Theorem; remarks about Stokes' Theorem, including a proof that Green's Theorem is a special case of Stokes' Theorem.

Lecture 42: (§17.8) CURL AND CIRCULATION DENSITY. Examples illustrating Stokes' Theorem; discussion of the physical interpretation of the curl as the direction of maximal circulation density in an incompressible, steady-state fluid flow.

Lecture 43: WHAT'S NEXT?. Brief discussion of the course(s) a student could/should take after completing Multivariable Integral Calculus.