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## Lecture Summaries for Multivariable Differential Calculus M52A

**Lecture 01** (§13.1) : Multivariable Differential Calculus. An introduction to the three dimensional Cartesian (or rectangular) coordinate system; discussion of the right-hand rule; coordinate representations of the coordinate axes, coordinate planes, and planes parallel to the coordinate planes; the distance formula in 3-space.

**Lecture 02** (§13.1) : Graphs in 3-space. An introduction to graphing equations in 3-space; discussion of equations of spheres and cylindrical surfaces.

**Lecture 03** (§13.2) : Vectors. An introduction to vectors; discussion of the geometric representation of vectors by directed line segments (or arrows) in 3-space; definitions of initial point, terminal point, equality of vectors, the zero vector, scalars; definition, discussion, and geometric representation of vector addition, subtraction, and multiplication by scalars.

**Lecture 04** (§13.2) : Coordinate Representations of Vectors. Coordinate representation of vectors in terms of components; proofs of the basic properties of vector algebra using coordinate representations.

**Lecture 05** (§13.2) : Norms, Unit Vectors, & Decomposition. Definitions of norm of a vector, unit vectors, normalization, linear combination, and the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ ; decomposition of a vector into a scalar times a unit vector; decomposition of a vector into a linear combination of the standard basis vectors.

**Lecture 06** (§13.2) : Vector Addition and Resultant Forces. Discussion, examples, and applications of the principle that if two forces, represented as vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, act on an object, then the two forces have the same effect as the single force represented by the vector $\mathbf{F}_1 + \mathbf{F}_2$.

**Lecture 07** (§13.3) : The Dot Product. Definition of dot product (or inner product); proof of the basic algebraic properties of dot products; proof that for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}\bullet\mathbf{v} = ||\mathbf{u}|| \cdot ||\mathbf{v}|| \cos\theta$, where $\theta$ is the angle between the vectors when drawn with common initial point; definition of orthogonal vectors.

**Lecture 08** (§13.3) : Direction Angles. Definition and discussion of direction angles and direction cosines.

**Lecture 09** (§13.3) : Orthogonal Projections. Proof that if $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors in 2-space, and $\mathbf{v}$ is a nonzero vector in 2-space, then $\mathbf{v} = (\mathbf{v}\bullet\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{v}\bullet\mathbf{e}_2)\mathbf{e}_2$; definitions of the vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, and the orthogonal projection, $\text{proj}_{\mathbf{u}}\mathbf{v}$, of $\mathbf{v}$ on a unit vector $\mathbf{u}$.

**Lecture 10** (§13.4) : Cross Products. Definition of the cross product; discussion of determinants and the relationship with cross products; proofs of the basic algebraic properties of cross product, including non-commutativity and non-associativity; computation and discussion of the results $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

**Lecture 11** (§13.4) : The Geometry of Cross Products. Proof that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors and $\theta$ is the angle between them, then $\mathbf{u}\times\mathbf{v}$ is the unique vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, has direction given by the right-hand rule, and has norm $||\mathbf{u}||\,||\mathbf{v}|| \sin\theta$; proof that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$; proof that the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has area $||\mathbf{u} \times \mathbf{v}||$.

**Lecture 12** (§13.4) : The Scalar Triple Product. Definition and discussion of the scalar triple product and its relationship to determinants; proof that $\mathbf{u}\bullet\mathbf{v}\times\mathbf{w} = \mathbf{w}\bullet\mathbf{u}\times\mathbf{v} = \mathbf{v}\bullet\mathbf{w}\times\mathbf{u}$ and $\mathbf{u}\bullet\mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v}\bullet\mathbf{w}$; proof that $|\mathbf{u}\bullet\mathbf{v} \times \mathbf{w}|$ gives the volume of the parallelepiped with adjacent sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$; proof that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ lie in the same plane if and only $\mathbf{u}\bullet\mathbf{v} \times \mathbf{w} = 0$; discussion of the coordinate independence of the dot product and cross product.

**Lecture 13** (§13.5) : Parametric and Vector Equations of Lines. Derivation of parametric equations and vector equations for lines in 3-space.

**Lecture 14** (§13.5) : Problems Involving Lines in 3-space. Various examples and problems using vector and parametric equations of lines in 3-space, including how to find vector and parametric equations for lines and line segments between two points, and how to determine if two lines intersect; definition of skew lines.

**Lecture 15** (§13.6) : Equations of Planes. Definitions of vector perpendicular to a plane, normal vector for a plane, plane parallel to a plane, and vector parallel to a plane; definition and derivation of point-normal equation for plane passing through a given point and having a given normal vector; definition of the general form of an equation for a plane, and proof that every equation of that form describes a plane; discussion of several examples and relevant existence and uniqueness questions.

**Lecture 16** (§13.6) : Planes: Angles and Distances. Definition of the acute angle of intersection between two intersecting planes; proof that the acute angle $\theta$ between intersecting planes with normals $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by $\cos\theta = |\mathbf{n}_1\bullet\mathbf{n}_2|/(||\mathbf{n}_1||\,||\mathbf{n}_2||)$; proof that the distance between the point $(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is $|ax_0 + by_0 + cz_0 + d|/\sqrt{a^2 + b^2 + c^2}$; proof that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $|d_1 - d_2|/\sqrt{a^2 + b^2 + c^2}$.

**Lecture 17** (§13.7) : Traces. Definition of the trace of a surface in a plane; discussion and examples showing how a surface can be reconstructed from various traces, and how to obtain trace curves from an equation defining the surface.

**Lecture 18** (§13.7) : Graphing Quadric Surfaces. Definition of a quadric surface; more examples of the use of the trace technique to construct graphs of second degree equations in $x$, $y$, and $z$; dicussion and example of the effect on the graph of an equation when $x$, $y$, and $z$ are replaced by $x - \alpha$, $y - \beta$, and $z - \gamma$ (where $\alpha$, $\beta$, and $\gamma$ are constants).

**Lecture 19** (§13.8) : Cylindrical Coordinates. Definition and discussion of cylindrical coordinates; graphs of equations of the form $r = r_0$, and $\theta = \theta_0$; conversion between cylindrical and rectangular coordinates.

**Lecture 20** (§13.8) : Spherical Coordinates. Definition and discussion of spherical coordinates; graphs of equations of the forms $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$; conversion between spherical coordinates and rectangular or cylindrical coordinates.

**Lecture 21** (§14.1) : Vector-valued Functions. Definition, discussion, and examples of vector-valued functions, their domains, graphs, and orientations.

**Lecture 22** (§14.2) : Calculus of Vector-valued Functions. Definitions of limit, continuity, differentiation, and integration of vector-valued functions; proofs of the basic properties of differentiation and integration, including the Fundamental Theorems of Calculus for vector-valued functions.

**Lecture 23** (§14.2) : GEOMETRY OF DERIVATIVES. Definition of the tangent vector, $\mathbf{r}'(t)$, and tangent line to a curve; proof that $\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$; geometric argument showing that $\mathbf{r}'(t)$, if nonzero, is indeed tangent to the graph of $\mathbf{r}$; proof of the "product rules" for differentiating dot products and cross product; proof that if the graph of $\mathbf{r}$ lies on a sphere centered at the origin, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are always orthogonal.

**Lecture 24** (§14.3) : CHANGES IN PARAMETRIZATIONS OF CURVES. Definition of a smooth vector-valued function; definition of smooth change of parameter; discussion of general method for constructing many parametrizations of a given curve; proof that if $\mathbf{r}_1$ is a smooth parametrization of $C$ and $g$ is a real-valued differentiable function then $\frac{d}{d\tau}(\mathbf{r}_1 \circ g)(\tau) = \mathbf{r}_1'(g(\tau)) \cdot g'(\tau)$, and if $g'$ is continuous and nonzero then the composition $\mathbf{r}_1 \circ g$ is a smooth parametrization of $C$; illustration of these results in various examples.

**Lecture 25** (§14.3) : PARAMETRIZATIONS IN TERMS OF ARC LENGTH. Review of arc length of a curve in 2-space; definition of arc length of a curve in 3-space; definition of a parametrization of a curve in terms of arc length ("arc length parametrization"); discussion showing how, using a smooth change of parameter, an arc length parametrization of a curve can be constructed from any given parametrization; proof that if $\mathbf{r}(s)$ is an arc length parametrization of a curve, then $||\mathbf{r}'(s)|| = 1$ for all $s$.

**Lecture 26** (§14.4) : UNIT TANGENTS, NORMALS, AND BINORMALS. Definitions, discussion, and examples of unit tangent vectors ($\mathbf{T}$), unit normal vectors ($\mathbf{N}$), and (for curves in 3-space) binormal vectors ($\mathbf{B}$); formulas for these vectors in terms of arc length parametrizations; definition of the **TNB** (Frenet) frame; proof that $\mathbf{B}$ can be expressed as $\mathbf{r}' \times \mathbf{r}''/||\mathbf{r}' \times \mathbf{r}''||$.

**Lecture 27** (§14.5) : CURVATURE. Informal discussion of curvature; formal definition and discussion of curvature, $\kappa$; proof that for any parametrization $\mathbf{r}(t)$ of a curve $C$, $\kappa(t = t_0) = \dfrac{||\mathbf{T}'(t_0)||}{||\mathbf{r}'(t_0)||}$, and $\kappa(t = t_0) = \dfrac{||\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)||}{||\mathbf{r}'(t_0)||^3}$.

**Lecture 28** (§14.5) : COMPUTATIONS OF CURVATURE. Computations of curvature for a line, circular helix, and general ellipse in 2-space, including computation showing that the curvature of a circle of radius $r$ is $1/r$; definition and discussion of osculating circle (circle of curvature) and radius of curvature for curves in 2-space; proof that for a curve in 2-space $\kappa = |\frac{d\phi}{ds}|$ where $\phi$ is the angle measured counterclockwise from the positive $x$ direction to the unit tangent vector, and $s$ is an arc length parameter.

**Lecture 29** (§14.6) : VECTOR CALCULUS AND MOTION ALONG A CURVE. Definitions and discussions of velocity ($\mathbf{v}$), speed ($ds/dt$), and acceleration ($\mathbf{a}$) of a particle moving along a curve; proof that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d^2s}{dt^2}$ and $a_N = \kappa \cdot (\frac{ds}{dt})^2$; example showing that for a particle traveling in a circle of radius $r$ at constant speed $v_0$, the acceleration points in the normal direction with magnitude $v_0^2/r$; example showing that for a particle moving along a straight line, the acceleration is parallel to the direction of motion and has magnitude equal to $|d^2s/dt^2|$; proof that $a_T$ and $a_N$ can be written: $a_T = \frac{\mathbf{v} \bullet \mathbf{a}}{||\mathbf{v}||}$, $a_N = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||}$.

**Lecture 30** (§14.6) : EXAMPLES AND PROJECTILE MOTION. Examples of results from previous lecture; observation that $||\mathbf{a}||^2 = |a_T|^2 + |a_N|^2$; discussion of projectile motion, including how Newton's Law gives the acceleration, and how to integrate acceleration to obtain the velocity and position functions; observation that a projectile travels in a parabolic path.

**Lecture 31** (§15.1) : MULTIVARIABLE FUNCTIONS. Notation for the set of all $n$-tuples of real numbers ($\mathbf{R}^n$); definition of a real-valued function of $n$ real variables; definition of its domain and range; definition, discussion, and examples of graphs of functions of two variables; definitions, discussions, and examples of level curves of functions of two variables, and level surfaces for functions of three variables.

**Lecture 32** (§15.2) : PROPERTIES OF SETS IN 2-SPACE AND 3-SPACE. Definitions, discussions, and examples of: ball of radius $r$ centered at $P$ ($B_r(P)$), interior point, boundary point, interior of a set, boundary of a set, open set, closed set, bounded set, unbounded set, accumulation point; brief review of limits for functions of one variable.

**Lecture 33** (§15.2) : LIMITS. Definitions, discussions, and examples of limit along a curve ($\lim_{(x,y)\underset{C}{\longrightarrow}(x_0,y_0)} f(x,y)$), and general limit ($\lim_{(x,y)\to(x_0,y_0)} f(x,y)$); definition of general limit for a function of three variables.

**Lecture 34** (§15.2) : CONTINUITY. Definition, discussion, and examples of continuity at a point for functions of two variables; definition of continuity on a set of points; definition of continuous function; statement and (partial) proof of theorem asserting that the sum, difference, product, and quotient of continuous functions is continuous; statement of theorem asserting that $f(x,y) = g(x)\cdot h(y)$ is continuous if $g$ and $h$ are continuous functions of one variable, and that $f(x,y) = g(h(x,y))$ is continuous if $g$ is a continuous function of one variable and $h$ is a continuous function of two variables.

**Lecture 35** (§15.3) : PARTIAL DERIVATIVES. General discussion of differentiation for a function of two variables; definitions, discussion, and examples of the partial derivatives $f_x$ and $f_y$ for a function of two variables.

**Lecture 36** (§15.3) : HIGHER-ORDER PARTIALS. Discussion of the second and higher-order partial derivatives of a function of two variables; discussion of "partial" ($\partial$) notation for partial differentiation; example computing higher-order partial derivatives and illustrating $\partial$ notation; statement of theorem asserting the equality of mixed partials; definition of partial derivative for a general real-valued of $n$ variables.

**Lecture 37** (§15.4) : DIFFERENTIABILITY. Example showing that existence of partial derivatives at a point does not imply continuity at that point; review, reformulation, and geometric interpretation of the notion of differentiability for a function of one variable; definition and discussion of differentiability for a function of two variables; proof that differentiability of $f$ at $(x_0, y_0)$ implies continuity of $f$ at $(x_0, y_0)$; statement of theorem giving sufficient conditions for differentiability in terms of the continuity of the first partials.

**Lecture 38** (§15.4) : CHAIN RULE. Review of statement and proof of the chain rule for functions of one variable; proof of the chain rule for functions of two variables: if $f$ is a differentiable function of $x$ and $y$, and $x$ and $y$ are differentiable functions of $t$, then $\frac{d}{dt}f(x(t),y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

**Lecture 39** (§15.4) : CHAIN RULE : EXAMPLES & CONSEQUENCES. Examples of the chain rule proven in previous lecture; discussion showing how the chain rule can be used as a substitute for implicit differentiation; proof and example of the chain rule in the case where $f$ is a function of $x$ and $y$, and both $x$ and $y$ are functions of $u$ and $v$.

**Lecture 40** (§15.5) : TANGENT PLANES. Definition of the tangent plane; proof of theorem giving the existence and formula for the tangent plane assuming $f$ is differentiable at the given point; definition and equation of normal line to a surface at a point; discussion of the defintion of differentiability and the resultant relationship between $f$ and its tangent plane at a point; definition of local linear approximation; definition and discussion of the total differential of $f$.

**Lecture 41** (§15.6) : DIRECTIONAL DERIVATIVES. Definition and notation for the directional derivative of $f$; Proof that the directional derivative of $f$ at $(x_0, y_0)$ in the direction of the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by $D_{\mathbf{v}} f(x_0, y_0) = (1/||\mathbf{v}||)(f_x(x_0, y_0)v_1 + f_y(x_0, y_0)v_2)$; proof of some basic properties of directional derivatives.

**Lecture 42** (§15.6) : THE GRADIENT. Definition and discussion of the gradient function $\nabla f$; proof that if $\mathbf{u}$ is a unit vector, then $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$; proof that $D_{\mathbf{u}} f(x, y)$ is maximal when $\mathbf{u}$ and $\nabla f(x, y)$ point in the same direction, and minimal when they point in opposite directions; proof that at $(x, y)$, $\nabla f$ is normal (orthogonal to a tangent vector) to the level curve of $f$ through $(x, y)$.

**Lecture 43** (§15.7) : FUNCTIONS OF $n$ VARIABLES: DIFFERENTIATION I. Definition of differentiability for a function of $n$ variables; discussion of necessary and sufficient conditions for differentiability; statement and example of the general chain rule for differentiable functions $f$ on $n$ variables $x_1, \ldots, x_n$, where $x_1, \ldots, x_n$ are differentiable functions of $m$ variables $t_1, \ldots, t_m$; definition and brief discussion of the total differential of a function of $n$ variables.

**Lecture 44** (§15.7) : FUNCTIONS OF $n$ VARIABLES: DIFFERENTIATION II. Definitions of directional derivatives and gradients for functions of an arbitrary number of variables; discussion of the basic relationship between the directional derivatives and gradient of such a function; proof that for functions of three variables, the gradient at any point is normal to the level surface through that point; derivation of the equation of the tangent plane to a level surface.

**Lecture 45** (§15.8) : EXTREMA OF FUNCTIONS OF TWO VARIABLES. Definitions of relative and absolute maxima, minima, and extrema; definitions of interior extrema and boundary extrema; statement of the Extreme-value Theorem for functions of two variables; definition of critical point; proof that every relative extremum of the function $f(x, y)$ occurs either at a boundary point of the domain or at a critical point.

**Lecture 46** (§15.8) : SECOND PARTIALS TEST. Statement of the Second Partials Test; applications and examples of the Second Partials Test.

**Lecture 47** (§15.9) : LAGRANGE MULTIPLIERS IN TWO DIMENSIONS. Definitions of constraint curve, constraint equation, and constrained extrema; discussion and example of the method of Lagrange Multipliers for functions of two variables; in particular, proof that for a function $f(x, y)$ and ("constraint") curve $C$ given by the equation $g(x, y) = 0$, all relative extrema of $f$ restricted to $C$ ("constrained relative extrema") must occur either at endpoints of $C$ or at points $(x, y)$ on the constraint curve where $\nabla f(x, y) = \lambda \nabla g(x, y)$, for some $\lambda$.

**Lecture 48** (§15.9) : LAGRANGE MULTIPLIERS IN THREE DIMENSIONS. Discussion and example of the method of Lagrange Multipliers for functions of three variables; in particular, proof that for a function $f(x, y, z)$ and ("constraint") surface $S$ given by the equation $g(x, y, z) = 0$, all relative extrema of $f|_S$ ("constrained relative extrema") must occur either at boundary points of $S$ or at points $(x, y, z)$ on the constraint curve where $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, for some $\lambda$.

**Lecture 49**: WHAT'S NEXT?. Brief discussion of the course(s) a student could/should take after completing Multivariable Differential Calculus.