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Lecture Summaries for Linear Algebra M51A

"§" refers to the section in the textbook.

Lecture 01 : Introduction. An introduction to the course, listing the topics to be covered; remarks concerning the rigor and formality with which the theory will be developed.

Lecture 02 (§1.1) : Matrices. Definitions, notations, and examples of: matrix, element, order, a_{ij}, (as well as A_{ij}, $A_{i,j}$, $A(i,j)$, i,j-entry of A), $[a_{ij}]$, row index, column index, main (or principal) diagonal, diagonal element, square matrix, row matrix, column matrix, components, dimension, n-tuple, equality of matrices.

Lecture 03 (§1.1) : Matrix Addition and Scalar Multiplication. Definition and example of matrix addition; theorem asserting commutativity and associativity of (matrix) addition; definition of the zero matrix; theorem asserting that $\mathbf{0}$ is an additive identity element; definition of matrix subtraction; definition and example of scalar multiplication; theorem asserting the basic associative and distributive properties of scalar multiplication; some basic properties (left as exercises) relating matrix addition, subtraction, and scalar multiplication.

Lecture 04 (§1.2) : Matrix Multiplication. Definition of matrix multiplication; examples showing that matrix multiplication is not commutative, that a product of two non-zero matrices can return the zero matrix, that "cancellation" does not hold for matrix multiplication; theorem asserting that matrix multiplication is associative and distributes over matrix addition.

Lecture 05 (§1.3) : Transpose and Symmetry. Definition, examples, and basic properties of the transpose operation; notation A^n for $n > 0$ and for $n = 0$; definitions, examples, and basic properties of symmetric, skew-symmetric, diagonal, and identity matrices.

Lecture 06 (§1.3) : Partitions and Special Forms of Matrices. Definitions and examples of submatrix, partitioned matrix, block, symmetrically partitioned matrix, diagonal block of a symmetrically partitioned matrix, zero row of a matrix, nonzero row of a matrix, row-reduced form, upper triangular, lower triangular; theorem asserting that the product of two upper triangular matrices is upper triangular, and the product of two lower triangular matrices is lower triangular.

Lecture 07 (§1.4) : Linear Systems of Equations. Definition and examples of a system of m linear equations in n variables, and its solutions; matrix representation of such a system; theorem asserting that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct solutions to $A\mathbf{x} = \mathbf{b}$, then $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ is also a solution whenever $\alpha + \beta = 1$; corollary asserting that a system of linear equations has either 0, 1, or infinitely many solutions; geometrical perspective of this corollary in 2-space and 3-space; definitions of homogeneous, inhomogeneous, consistent, inconsistent, and trivial solution; observation that the trivial solution makes every homogeneous system consistent.

Lecture 08 (§1.4) : Gaussian Elimination: Part I. Definitions and examples of augmented matrix; definition of elementary row operations; correspondence between operations on equations of a linear system and the elementary row operations on the corresponding augmented matrices; definition and example of Gaussian elimination; definition of derived set (of equations); guidelines for applying elementary row operations when doing Gaussian elimination.

Lecture 09 (§1.4) : GAUSSIAN ELIMINATION: PART II. More examples of Gaussian elimination, including an example of an inconsistent system and a system with infinitely many solutions; summary of the relationship between the derived set of equations and the number of solutions to the system.

Lecture 10 (§1.5) : THE INVERSE OF A MATRIX. Definition and examples of (multiplicative) inverse of a (square) matrix; observation that being an inverse is a symmetric relation; theorem asserting that inverses are unique; A^{-1} notation for the inverse; observation that not all matrices have inverses; definitions of invertible, nonsingular, and singular; observation that diagonal matrices with diagonal elements all nonzero are invertible; theorem asserting that $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$, and $(\lambda A)^{-1} = (1/\lambda)A^{-1}$, for any nonzero scalar λ; theorem asserting that if A is invertible, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Lecture 11 (§1.5) : ELEMENTARY MATRICES AND COMPUTING INVERSES. Definition and example of elementary matrix; construction of elementary matrices; theorem asserting elementary matrices are unique and invertible, and giving descriptions of the inverses; lemmas asserting the equivalence of invertibility of a matrix A with A's having certain row-reduced forms; theorem asserting that A is invertible if and only if $E_k \cdots E_1 A = I$, for some elementary matrices $E_1, \ldots, E_k$; step-by-step procedure for computing the inverse of A using the augmented matrix $[A|I]$ and the elementary matrices (or row operations) that transform A into the identity (if such exist).

Lecture 12 (§1.6) : LU DECOMPOSITIONS. Definition of LU decomposition of a nonsingular matrix; notation $R_3(i,j,k)$ denoting the type three elementary row operation that adds to the ith row k times the jth row (where $k \neq 0$ and $i \neq j$); lemma asserting that the diagonal elements of a nonsingular lower triangular matrix must be nonzero; theorem asserting that the nonsingular matrix A has an LU decomposition if and only if A can be transformed to an upper triangular matrix using only operations of the form $R_3(i,j,k)$ with $i > j$; discussion of LDU decomposition; discussion of how such decompositions can be used to solve systems of equations.

Lecture 13 (§2.1) : VECTOR SPACES. Notations $\mathbf{R}$, $\mathbf{C}$, and ϵ; definition (and defining axioms) of a vector space; definition of a real vector space and a complex vector space; example showing that $\mathbf{R}^3$, with vector addition and scalar multiplication defined componentwise, is a real vector space.

Lecture 14 (§2.1) : EXAMPLES OF VECTOR SPACES. Discussion of various examples of vector spaces including $\mathbf{R}^n$, $\mathbf{C}^n$, certain sets of matrices, polynomials, real-valued functions, and solutions to linear homogeneous differential equations.

Lecture 15 (§2.1) : BASIC PROPERTIES OF VECTOR SPACES. Theorems asserting that in any vector space there is a unique additive identity element (zero vector), additive inverses are unique, $0 \odot \mathbf{u} = \mathbf{0}$, $(-1) \odot \mathbf{u} = -\mathbf{u}$, $-(-\mathbf{u}) = \mathbf{u}$, $\alpha \odot \mathbf{0} = \mathbf{0}$, and $\alpha \odot \mathbf{u} = \mathbf{0}$ implies either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Lecture 16 (§2.2) : SUBSPACES: PART I. Definition of a subspace of a vector space; theorem asserting that a nonempty subset S of a vector space V is a subspace of V if and only if S is closed under vector addition and scalar multiplication; corollary asserting that a nonempty subset of a vector space V is a subspace of V if and only if $\alpha \odot \mathbf{u} \oplus \beta \odot \mathbf{v} \in S$, whenever $\mathbf{u}, \mathbf{v} \in S$ and α and β are scalars; examples illustrating the use of the corollary.

Lecture 17 (§2.2) : SUBSPACES: PART II. Example showing that for any $a_1, \ldots, a_n \in \mathbf{R}$, the set $S = \{(x_1, \ldots, x_n \in \mathbf{R}^n \;\big|\; a_1x_1 + \cdots + a_nx_n = 0\}$ is a subspace of $\mathbf{R}^n$; discussion of the geometry of such subspaces; theorem asserting that the intersection of arbitrarily many subspaces is again a subspace; definitions of linear combination and span of a collection of vectors; theorems asserting that for any nonempty subset S of a vector space V, $span(S)$ is a subspace of V, $span(S)$ is contained in any subspace containing S, and $span(S)$ is the intersection of all subspaces containing S.

Lecture 18 (§2.3) : Linear Independence. Definitions and examples of linear dependence and independence; theorem asserting that a finite ordered set of nonzero vectors is linearly dependent if and only if one of the vectors is a linear combination of those preceding it; corollary asserting that if S is a linearly dependent subset of V having n elements and with $V = span(S)$, then there is a subset T of S with $n-1$ elements such that $span(T) = V$; corollary asserting that if $V = span(S)$ and S has the minimal number of elements of all sets whose span is V, then S is linearly independent; theorems asserting that a subset with one element $\mathbf{u}$ is linearly dependent if and only if $\mathbf{u} = \mathbf{0}$, a subset with two elements is linearly dependent if and only if one element is a scalar multiple of the other, and any subset containing $\mathbf{0}$ is linearly dependent; theorem asserting that a subset of a linearly independent set is linearly independent, and a superset of a linearly dependent set is linearly dependent.

Lecture 19 (§2.4) : Basis. Definitions of spans, spanning set, and basis; various examples of bases including the standard basis for $\mathbf{R}^n$ and the standard basis for P^n; definition of finite dimensional; examples of finite dimensional vector spaces; examples of infinite dimensional vector spaces; theorem asserting that if S has n elements and spans V, then any set with more than n elements must be linearly dependent.

Lecture 20 (§2.4) : Dimension. Various corollaries of the theorem proven in the previous lecture, including the corollary asserting that for a given finite dimensional vector space, all bases have the same number of elements; definition, notation, and examples of dimension; remark on how dimension can change if the set of scalars is changed; remark on the dimension of the trivial vector space $\{\mathbf{0}\}$ being zero; theorem asserting that for a given basis of a vector space V, each $\mathbf{u} \in V$ has a unique representation as a linear combination of the basis vectors; definition, notation, and examples of coordinates of a vector with respect to a basis; theorem asserting that any linearly independent set of vectors is a subset of some basis, and that any spanning set contains some basis as a subset.

Lecture 21 (§2.5) : Row Space : Part I. Definitions, notations and examples of row space and row rank of a matrix; theorem asserting that a basis for the row space of a matrix in row-reduced form is the set of nonzero rows, and the row rank is the number of nonzero rows; lemmas asserting that $S \subset T$ implies $span(S) \subset span(T)$, $span(span(S)) = span(S)$, and $S \subset span(T)$ implies $span(S) \subset span(T)$; lemma asserting that the span of a set of nonzero vectors is unchanged if (1) a multiple of one vector is added to another, and (2) one of the vectors is replaced by a nonzero scalar multiple of itself; theorem asserting that if matrix B is obtained from matrix A by elementary row operations, then $rowspace(B) = rowspace(A)$.

Lecture 22 (§2.5) : Row Space : Part II. Corollary (of theorems from the previous lecture) asserting that a basis for $rowspace(A)$ is the set of nonzero rows of matrix B obtained by transforming A to row-reduced form; example illustrating the use of this corollary in finding a basis for the span of any finite collection of n-tuples in $\mathbf{R}^n$; discussion explaining how a similar procedure can be used to find a basis for the span of any finite collection of vectors in any vector space V; lemma and theorem asserting that in any n-dimensional vector space V with some chosen basis, a basis for the span of the coordinate representations of $\mathbf{u}_1, \ldots, \mathbf{u}_m$ consists of n-tuples that are the coordinate representations of vectors that form a basis for $span\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$.

Lecture 23 (§2.6) : Column Space and Rank of a Matrix. Definitions and notations for column space and column rank of a matrix; lemma and theorem asserting that row rank equals column rank; definition and notation for rank of a matrix; theorem asserting that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $rank(A) = rank([A|\mathbf{b}])$; theorem asserting that if $A\mathbf{x} = \mathbf{b}$ is a consistent system of n variables and if $rank(A) = k$, then the solutions to the system are expressible in terms of $n - k$ arbitrary unknowns; corollary asserting that the homogeneous system $A\mathbf{x} = \mathbf{0}$ in n variables has nontrivial solutions if and only if $rank(A) \neq n$.

Lecture 24 (§2.6) : RANK AND INVERTIBILITY. Lemma asserting that if A and B are $n \times n$ matrices satisfying $AB = I_n$, then the rows of A are linearly independent and $r(A) = n$; lemma asserting that if an $n \times n$ matrix A has rank n, there there exists C such that $CA = I_n$; lemma asserting that if A and B are $n \times n$ matrices with $AB = I_n$, then $BA = I_n$; theorem asserting that an $n \times n$ matrix A is invertible if and only if $r(A) = n$; corollary asserting that an $n \times n$ matrix A is invertible if and only if it can be transformed using elementary row operations to an upper triangular matrix with all diagonal elements equal to 1.

Lecture 25 (§3.1,3.2) : LINEAR TRANSFORMATIONS. Definitions of function, transformation, image, one-to-one, onto, linear; discussion of what "linear" really means; various examples of linear transformations; theorem asserting that if $T : \mathbf{R}^m \to \mathbf{R}^n$ is defined by $T\mathbf{v} = A\mathbf{v}$ where A is an $n \times m$ matrix, then T is linear.

Lecture 26 (§3.3) : MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS. Discussion of why a linear transformation is determined by its action on a basis of its domain; notation $(\mathbf{v})_B$ denoting the column matrix consisting of the coordinates of $\mathbf{v}$ with respect to the basis B; lemma asserting that taking coordinates with respect to a basis of an n-dimensional vector space V gives a one-to-one, onto, linear transformation from V to $\mathbf{R}^n$; theorem asserting that if T is a linear transformation from the vector space V to the vector space W with bases B and C respectively, then there exists a matrix A_B^C such that for any $\mathbf{v} \in V$, $(T\mathbf{v})_C = A(\mathbf{v})_B$; pictorial description of the previous theorem.

Lecture 27 (§3.4) : MATRIX REPRESENTATIONS AND CHANGE OF BASIS. Corollary (of theorem proven in previous lecture) asserting that for any two bases B and C of the finite dimensional vector space V, there exists a transition matrix P_B^C such that $(\mathbf{v})_C = P_B^C(\mathbf{v})_B$ for all $\mathbf{v} \in V$; theorem asserting that any transition matrix P_B^C is invertible and $(P_B^C)^{-1} = P_C^B$; definition of similar; theorem asserting that for bases B and C of V, and linear transformation $T : V \to V$, the matrix representations of T with respect to bases B and C are similar; theorem asserting that any matrix similar to a matrix representation of T (with respect to some basis) is also a matrix representation of T (with respect to some basis).

Lecture 28 (§3.5) : KERNEL AND IMAGE : PART I. Definitions, notations, and examples or kernel, image, nullity, and rank of a linear transformation; theorems asserting that the kernel and image are subspaces (of the domain and range respectively); theorem asserting that if A is the matrix representation of $T : V \to W$ with respect to bases for V and W, then the rank of T is the rank of A.

Lecture 29 (§3.5) : KERNEL AND IMAGE : PART II. Lemma asserting that if $T : V \to W$ is linear, and if $ker(T)$ has a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that can be extended to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for V, then $\{T\mathbf{v}_{k+1}, \ldots, T\mathbf{v}_n\}$ is a basis for $image(T)$; theorem asserting that if $T : V \to W$ is linear and V is finite dimensional, then $r(T) + null(T) = dim(V)$; theorem asserting that a linear transformation T is one-to-one if and only if $ker(T) = \{\mathbf{0}\}$; theorem asserting that if $T : V \to W$ is linear and $dim(V) = dim(W)$, then T is one-to-one if and only if T is onto; definition of an isomorphism; definition and notation ($\cong$) of isomorphic; theorem asserting that $\cong$ is a reflexive, symmetric, and transitive relation; definition of equivalence relation; theorem asserting that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

Lecture 30 (§4.1, 4.2) : Determinants: Part I. Notation $\mu_{ij}(A)$ denoting the submatrix of A obtained by deleting the ith row and jth column of A; definition, notation, and examples of the determinant of a matrix; definition of minor and cofactor; theorem of expansion by cofactors; corollary asserting that a matrix with a zero row or zero column has determinant zero; theorem asserting that the determinant of an upper (or lower) triangular matrix is the product of the diagonal elements; statement of two theorems relating 2×2 determinants to areas of parallelograms in the plane.

Lecture 31 (§4.2) : Determinants: Part II. Theorem asserting that for any square matrix A, $\det(A) = \det(A^T)$; three lemmas that together imply a theorem asserting that for any $n \times n$ matrix A and $n \times n$ elementary matrix E, $\det(EA) = \det(E)\det(A)$; corollary asserting that $\det(A) = 0$ whenever A has two identical rows or two identical columns; corollary asserting that for any scalar λ and any $n \times n$ matrix A, $\det(\lambda A) = \lambda^n \det(A)$.

Lecture 32 (§4.2) : Determinants: Part III. Corollaries (of theorem in previous lecture) asserting that for any $n \times n$ matrix A and any $n \times n$ elementary matrices $E_1, \ldots, E_k$, $\det(E_k E_{k-1} \cdots E_1 A) = \det(E_k E_{k-1} \cdots E_1)\det(A) = \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)$; theorem asserting that a square matrix A is invertible if and only if $\det(A) \neq 0$; theorem asserting that for any $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$; corollary asserting that for an $n \times n$ invertible matrix A, $\det(A^{-1}) = (\det(A))^{-1}$; theorem asserting that similar matrices have the same determinant.

Lecture 33 (§4.3) : Eigenvectors and Eigenvalues. Definitions and examples of eigenvectors and eigenvalues for a transformation and for a matrix; theorem asserting that for the linear transformation $T : V \to V$, there exists a basis of V containing only eigenvectors of T if and only if there exists a matrix representation of T that is diagonal; theorem asserting that $\mathbf{u}$ is an eigenvector with eigenvalue λ for the linear transformation T if and only if the coordinates of $\mathbf{u}$ with respect to basis C is an eigenvector with eigenvalue λ for the matrix representation of T with respect to C; theorem asserting that for a linear transformation $T : V \to V$, the set S_λ, consisting of $\mathbf{0}$ and all eigenvectors of T with eigenvalue λ, is a subspace of V, known as the eigenspace of T for eigenvalue λ.

Lecture 34 (§4.3) : Finding Eigenvectors and Eigenvalues. Theorem asserting that λ is an eigenvalue of the $n \times n$ matrix A if and only if $\det(A - \lambda I_n) = 0$; examples illustrating how the theorem is used to find eigenvalues, eigenvectors, and eigenspaces.

Lecture 35 (§4.4) : Properties of Eigenvectors and Eigenvalues. Theorem asserting that similar matrices have identical characteristic equations (although the converse is false in general); theorem asserting that the eigenvalues of a triangular matrix are the diagonal elements; theorem asserting that a matrix is singular if and only if it has a zero eigenvalue; corollary giving various equivalences of invertibility, including having all eigenvalues be nonzero; theorem asserting that if $\mathbf{x}$ is an eigenvector with eigenvalue λ for the invertible matrix A, then $\mathbf{x}$ is an eigenvector with eigenvalue $1/\lambda$ for A^{-1}; theorem asserting that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then for each scalar k, $\mathbf{x}$ is an eigenvector of kA with eigenvalue $k\lambda$, and for each positive integer n, $\mathbf{x}$ is an eigenvector of A^n with eigenvalue λ^n; definition of trace; theorem asserting that the trace is the sum of the eigenvalues, and the determinant is the product of the eigenvalues.

Lecture 36 (§4.5) : Diagonalization : Part I. Definition of diagonalizable; theorem asserting that an $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors; theorem asserting that if A is similar to the diagonal matrices D_1 and D_2, then D_1 and D_2 have the same set of diagonal elements and each such element occurs with the same multiplicity in D_1 as it does in D_2.

Lecture 37 (§4.5) : DIAGONALIZATION : PART II. Theorem asserting that any set of eigenvectors having distinct eigenvalues must be linearly independent; corollary asserting that if A is $n \times n$ and A has n distinct eigenvalues, then A is diagonalizable; theorem asserting that if $\lambda_1, \ldots, \lambda_k$ are the (real) eigenvalues of the $n \times n$ matrix A, then A is diagonalizable if and only if the sum of the dimensions of the corresponding eigenspaces is n; theorem asserting that if λ is an eigenvalue of the $n \times n$ matrix A, then $dim(S_\lambda) = n - r(A - \lambda I_n)$.

Lecture 38 : WHAT'S NEXT?. Brief discussion of the course(s) a student could/should take after completing Linear Algebra.