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Lecture Summaries for Real Analysis M115

Lecture 01 (§2,3): Real Analysis. An introduction to the course; discussion of the axioms for an ordered field; proof of the rational roots theorem involving possible rational roots of integer polynomials; corollary that there are many real numbers which are not rational.

Lecture 02 (§4): The Completeness Axiom. Definition and examples of maximum and minimum (of a set), concepts of bounded above, bounded below, upper bound, lower bound, bounded, supremum (sup) and infimum (inf) (of a set); discussion of the completeness axiom; proof that every non-empty subset of reals which is bounded below has a greatest lower bound; proof that the rationals are not complete.

Lecture 03 (§4): Consequences of Completeness. Proof of some basic consequences of completeness, including the Archimedean property and the fact that the rationals are dense in the reals; brief discussion of guidelines for what is acceptable to use in a proof.

Lecture 04 (§5): $+\infty$ and $-\infty$. Definition of the symbols $+\infty$ and $-\infty$; discussion of what these symbols mean and how they can be used; extension of the definitions of sup and inf to any non-empty subset of reals.

Lecture 05 (§7): Sequences and Limits. Definition of and notations for sequences; proof of some basic properties of the absolute value function; definition and discussion of the limit of a sequence.

Lecture 06 (§7,8): Proofs Involving Limits. Specific examples of proofs involving limits of sequences; proof that limits of sequences are unique; proof of the squeeze theorem for sequences.

Lecture 07 (§9): Basic Limit Theorems for Sequences. Proof that the limit of a sum is the sum of limits; proofs for analogous statements involving differences, products, and quotients; proof that the limit of a constant times a sequence is that constant times the limit of the sequence; proof that convergent sequences are bounded.

Lecture 08 (§9): Limits of Some Basic Sequences. Proofs of the following: $\lim(1/n^p) = 0$ for $p > 0$; $\lim a^n = 0$ for $|a| < 1$; $\lim n^{1/n} = 1$; $\lim a^{1/n} = 1$ for $a > 0$. Proof that the limit of the quotient of two polynomials of the same degree is the ratio of their leading coefficients. Proof of the first part of the ratio test.

Lecture 09 (§9): Limits Equaling Infinity. Definition of statements of the form $\lim s_n = +\infty$ and $\lim s_n = -\infty$; proof that if $s_n \leq t_n$ for all n, then $\lim s_n = +\infty$ implies $\lim t_n = +\infty$, and $\lim t_n = -\infty$ implies $\lim s_n = -\infty$; proof that if $\lim s_n = +\infty$ and $\lim t_n > 0$, then $\lim s_n t_n = +\infty$; proof that $\lim s_n = +\infty$ if and only if $\lim(1/s_n) = 0$; proof that $\lim |p(n)/q(n)| = +\infty$ if p and q are polynomials with degree of p greater than degree of q; proof that $\lim s_n = +\infty$ if $s_n > 0$ and $\lim |s_{n+1}/s_n| > 1$.

Lecture 10 (§10): Bounded Monotone Sequences. Definitions of non-decreasing, non-increasing, and monotone; proof that a monotone sequence has a limit, and that this limit is finite if and only if the sequence is bounded; definition and discussion of lim sup and lim inf; proof that $\lim s_n = s$ if and only if $\liminf s_n = \limsup s_n = s$.

Lecture 11 (§10): Cauchy Sequences. Definition of a Cauchy sequence; proof that Cauchy sequences are bounded; proof that a sequence converges if and only if it's Cauchy.

Lecture 12 (§11): Subsequences. Definition of a subsequence; proof that if a sequence converges to s, then so does every subsequence; proof that s is a limit of a subsequence if and only if every interval around s contains infinitely many sequence elements; proof that every real number is the limit of a subsequence of any sequence containing all rationals; proof that there exists a subsequence of (s_n) converging to $\limsup s_n$, and similarly for $\liminf s_n$.

Lecture 13 (§11): More on Subsequences. Proof that every sequence has a monotone subsequence; proof that every bounded sequence has a convergent subsequence (Bolzano-Weierstrass); definition of subsequential limit; proof that s is a subsequential limit of (s_n) if and only if there's a monotone subsequence converging to s; proof of the basic properties of the set S of subsequential limits of (s_n), including $S \neq \emptyset$, $\limsup s_n = \sup S$, and $\liminf s_n = \inf S$; definition of a closed set; proof that S is closed.

Lecture 14 (§12): More on Limsups and Liminfs. Proof that if (s_n) converges to $s > 0$, then $\limsup s_n t_n = s \cdot \limsup t_n$; proof that for a positive sequence (s_n), $\liminf |s_{n+1}/s_n| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup |s_{n+1}/s_n|$.

Lecture 15 (§14): Introduction to Series. Notation for and definition of an infinite series; definition of the sequence of partial sums; definitions of convergence, divergence, and absolute convergence of an infinite series; proof that if a series converges then the sequence of terms converges to 0.

Lecture 16 (§14): Geometric Series and p-series. Definition and discussion of geometric series and p-series; proof that the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $a/(1-r)$ if and only if $|r| < 1$ or $a = 0$; proof that the harmonic series $\sum 1/n$ diverges; proof that $\sum 1/n^p$ diverges for $p \leq 1$.

Lecture 17 (§14): Tests for Series Convergence. Definition and discussion of the Cauchy criterion for convergence of a series; proofs of the comparison, root, and ratio tests for series.

Lecture 18 (§15): The Integral and Alternating Series Tests. Proof of the integral test; proof that p-series converge for $p > 1$; proof of the alternating series test; definition of conditional convergence.

Lecture 19 (§17): Continuity: Part I. Review of definition of a real-valued function of a real variable; review of standard terminology used when dealing with functions; the ϵ-δ definition of continuity at a point; the sequence definition of continuity at a point; proof that the two definitions of continuity are equivalent.

Lecture 20 (§17): Continuity: Part II. Discussion of the basic ways functions can be combined to create new functions; proofs that if f and g are continuous, then so are the functions $af+bg$, fg, f/g, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$; discussion of composition; proof that if g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

Lecture 21 (§18): The Extreme Value and Intermediate Value Theorems. Proof that a function, defined and continuous on a closed and bounded interval, is bounded; proof of the Extreme Value Theorem; proof of the Intermediate Value Theorem; proof that continuous functions map intervals to intervals.

Lecture 22 (§18): CONSEQUENCES OF THE INTERMEDIATE VALUE THEOREM. Consequences of the Intermediate Value Theorem including: proof that if f maps $[a, b]$ into itself and f is continuous, then $f(x) = x$ for some $x \in [a, b]$; proof that for every integer $m > 0$, every non-negative number has a non-negative mth root.

Lecture 23 (§18): INTERMEDIATE VALUE THEOREM AND INVERSE FUNCTIONS. Definitions of one-to-one, and inverse function; proof that a one-to-one continuous function defined on an interval must be either strictly increasing or strictly decreasing; proof that a strictly increasing (or decreasing) function mapping an interval to an interval must be continuous; proof that if f is one-to-one and continuous on some interval, then f^{-1} is also continuous.

Lecture 24 (§19): UNIFORM CONTINUITY. Definition and discussion of uniform continuity; proof that the function $1/x$ is not uniformly continuous on $(0, \infty)$, but is uniformly continuous on $[a, \infty)$, where $a > 0$; proof that linear functions are uniformly continuous on the whole real line; proof that the function 2^x is not uniformly continuous on $[0, \infty)$.

Lecture 25 (§19): PROPERTIES OF UNIFORMLY CONTINUOUS FUNCTIONS. Proof that a continuous function on a closed and bounded interval is uniformly continuous (on that interval); proof that if f is uniformly continuous on a set S then the image of every Cauchy sequence in S is Cauchy; proof that f is uniformly continuous on (a, b) if and only if there exists a continuous extension of f to $[a, b]$; proof that if f is differentiable on an interval I and f' is bounded, then f is uniformly continuous on I.

Lecture 26 (§20): LIMITS OF FUNCTIONS. Definition of the limit of a function along a subset of its domain; definitions of right-hand limit, left-hand limit, two-sided limit, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$; examples involving computations of such limits; proof that the limit of a sum, product, and quotient of functions is the sum, product, and quotient of the limits; proof that if $\lim_{x\to a^S} f(x) = L$ and g is continuous at L, then $\lim_{x\to a^S} g(f(x)) = g(L)$.

Lecture 27 (§20): ϵ-δ CHARACTERIZATIONS OF LIMITS OF FUNCTIONS. Proofs of various theorems characterizing limit statements using ϵ-δ terminology; proof that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

Lecture 28 (§23): INTRODUCTION TO POWER SERIES. Definition of power series; proof that $\sum a_n(x - x_0)^n$ converges for $|x - x_0| < 1/\beta$, where $\beta = \limsup |a_n|^{1/n}$; definition of radius of convergence and interval of convergence of a power series.

Lecture 29 (§23): EXAMPLES OF POWER SERIES. Some examples of power series and the computation of their intervals of convergence.

Lecture 30 (§24): POINTWISE CONVERGENCE AND UNIFORM CONVERGENCE. Definition and discussion of pointwise convergence of functions; definition and discussion of uniform convergence of functions; examples to illustrate both types and their differences; proof that continuous functions converging uniformly converge to a continuous function; sketch of argument showing that a power series is continuous on its interval of convergence.

Lecture 31 (§25): UNIFORM CONVERGENCE AND UNIFORMLY CAUCHY. Discussion of uniform convergence and commuting limits with other operations; proof that if $f_n \to f$ uniformly on $[a, b]$, then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$; definition of uniformly Cauchy; proof that a sequence of functions which is uniformly Cauchy converges uniformly.

Lecture 32 (§25,26): UNIFORM CONVERGENCE OF SERIES. Definitions of a series of functions, convergence and uniform convergence of a series of functions, uniform Cauchy criterion;

proof of the Weierstrass M-test; proof that if $\sum g_k$ converges uniformly then the sequence (g_k) converges uniformly to 0; proof that a function defined by a power series is continuous at any point lying strictly inside its interval of convergence.

Lecture 33 (§26): DIFFERENTIATION AND INTEGRATION OF POWER SERIES. Proofs that within its interval of convergence a power series can be integrated and differentiated term by term; application of the previous results to the computation of $\sum_{n=1}^{\infty} n/2^n$.

Lecture 34 (§26): ABEL'S THEOREM. Proof of Abel's theorem that if a series converges at an endpoint of its interval of convergence, then it's continuous at that endpoint; proof that the alternating series sums to $\ln 2$.

Lecture 35 (§28): THE DERIVATIVE. Definition of the derivative; computations of the derivatives of x^n and $x^{1/n}$ for n a positive integer; proof that the derivative of a linear combination of functions is that linear combination of the derivatives.

Lecture 36 (§28): PRODUCTS, QUOTIENTS, AND COMPOSITIONS. Proofs of the product, quotient, and chain rules for differentiation.

Lecture 37 (§29): MEAN VALUE THEOREM. Proof that if f is differentiable on (a,b), $x_0 \in (a,b)$, and $f(x_0)$ is either a max or a min, then $f'(x_0) = 0$; proof of Rolle's theorem; proof of the Mean Value theorem; proof that $f' = 0$ throughout an open interval implies f is constant on that interval; proof that any two functions which have the same derivatives throughout an interval must differ only by a constant.

Lecture 38 (§29): CONSEQUENCES OF THE MEAN VALUE THEOREM AND DERIVATIVES OF INVERSES. Proof that $f' > 0$ on (a,b) implies f is strictly increasing on (a,b); proofs of similar statements for $f' < 0$, $f' \geq 0$, and $f' \leq 0$; proof of the Intermediate Value theorem for derivatives; proof that if f is one-to-one, continuous in an interval containing x_0, and $f'(x_0)$ exists but is nonzero, then $(f^{-1})'(y_0) = 1/f'(f^{-1}(y_0))$.

Lecture 39 (§30): L'HOSPITAL'S RULE. Proof of L'Hospital's rule.

Lecture 40 (§31): TAYLOR SERIES. Definition of Taylor series; proof that series representations are unique; definition of the remainder function R_n; proof of Taylor's Theorem describing the remainder function; proof that if the derivatives of f are uniformly bounded, then the Taylor series for f represents f; computation of the Taylor series for e^x, and proof that the Taylor series converges to e^x for all $x \in \mathbf{R}$; similar computation and proof for the function $\cos x$.

Lecture 41 (§31): MORE ON TAYLOR SERIES. Proof of the integral version of Taylor's Theorem; proof of Cauchy's form of the remainder; proof that the function $f(x) = e^{-1/x}$ for $x > 0$, and $f(x) = 0$ for $x \leq 0$, has $f^{(n)}(0) = 0$ for all n, so its Taylor series is the 0 series, which does not converge to f on any open interval containing 0.