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Lecture Summaries for Complex Analysis M106

Lecture 01: An introduction to the complex number system; preliminary definitions, terminology, notations; discussion of the basic algebraic properties of complex numbers.

Lecture 02: The representation of complex numbers as vectors; the operations of *conjugation* and absolute value, and their relationships with sums and products.

Lecture 03: The polar representation of complex numbers; an introduction to *arguments* and exponential notation; discussion of the geometry of complex multiplication.

Lecture 04: Finding and expressing the nth roots of complex numbers.

Lecture 05: An introduction to the basic concepts and terminology associated with subsets of complex numbers; definitions of *neighborhoods, interior, exterior, boundary*, and *accumulation points, open, closed, closure, connected, bounded, domain, and region.*

Lecture 06: Basic definitions and notations for functions of a complex variable; elementary techniques for graphing these functions.

Lecture 07: Definition of *limit* for functions of a complex variable; basic theorems for limits of sums, products, and quotients; example of a function without a limit.

Lecture 08: Definitions of limit as z approaches *infinity* and of limits equaling infinity; discussion of the *extended complex plane, stereographic projection*, and the *Riemann sphere*.

Lecture 09: Definition of *continuity* and the basic properties of continuous functions.

Lecture 10: Definition of the *derivative*; basic properties of and formulas for differentiation.

Lecture 11: Derivation of the *Cauchy-Riemann equations*; examples of their use.

Lecture 12: Definitions of *analyticity*; basic facts about analytic functions; definition of *singular point*.

Lecture 13: Definition of *harmonic*; relationship between harmonic functions and real parts of analytic functions; definition and basic properties of *harmonic conjugates*; technique for finding harmonic conjugates.

Lecture 14: Definition and basic properties of the *complex exponential function*.

Lecture 15: Definitions and basic properties of the *complex trigonometric functions*.

Lecture 16: Definition and basic properties of the multi-valued complex *log* function; discussion of single-valued *branches* of the log function, their domains of definition, *branch cuts*, and analyticity properties.

Lecture 17: Discussion of the basic logarithmic identities involving the log of a product, log of a quotient, and log of a power.

Lecture 18: Definition of complex exponentiation with any base and any exponent; examination of the basic properties of the corresponding multi-valued functions and their single-valued branches.

Lecture 19: Basic differential and integral properties of complex-valued functions of a real variable, including the fundamental theorem of calculus and the relationship between the absolute value of an integral and the integral of the absolute value; definition of *piecewise continuous*.

Lecture 20: Definitions of *arc, simple arc, simple closed curve, differentiable arc, smooth arc, contour, simple closed contour*; discussion of length of an arc, and the Jordan Curve theorem.

Lecture 21: Definition and basic properties of *contour integration*, including independence of parametrizations, and upper bound estimates in terms of maximum value of function and length of contour.

Lecture 22: Computations of some sample contour integrals, and an illustration of the technique used for estimating bounds of contour integrals.

Lecture 23: Definition of *antiderivative*; proof of the equivalence of independence of path for contour integrals and the existence of an antiderivative.

Lecture 24: Statement and proof of the Cauchy-Goursat theorem; definitions of *simply connected* and *multiply connected*; generalizations of the Cauchy-Goursat theorem to mutliply connected domains; discussion of the "Principle of deformation of paths."

Lecture 25: Discussion and proof of the Cauchy Integral Formula.

Lecture 26: Discussion and proof of the generalized Cauchy Integral Formula, giving integral representations for the derivatives of an analytic function.

Lecture 27: Discussion and proof of the basic corollaries of the generalized Cauchy Integral Formula: analytic functions are infinitely differentiable; real and imaginary parts of analytic functions have continuous partials of all orders; a function is analytic if integrals along all closed contours in a domain are 0.

Lecture 28: Discussion and proof of Cauchy's Inequality, Liouville's theorem, and the fundamental theorem of algebra.

Lecture 29: Discussion and proof of theorems describing where analytic functions can attain their maxima; in particular, proof of the Maximum Modulus Principle.

Lecture 30: Definitions, terminology, and basic properties of complex sequences and series. In particular, definitions of: *convergence* (for sequences and series), *partial sum of a series*, *remainder of a series*, *absolute convergence*, and *power series*.

Lecture 31: Definition of Taylor series and Maclaurin series; proof that an analytic function is represented by its Taylor series.

Lecture 32: Maclaurin series for the exponential and trig functions, as well as the function $1/(1-z)$; examples involving computation of Taylor series for other slightly more general functions.

Lecture 33: Definition of *Laurent series*, and proof that a function analytic in an annular domain can be represented as a Laurent series throughout that domain.

Lecture 34: Computations of Laurent series for the functions $(1/z^2)\sin z$, $e^z/(z-1)^3$, and $1/(z^2-z^3)$.

Lecture 35: Definitions of *pointwise convergence* and *uniform convergence* of sequences and series of functions; examples of each; proof that continuous functions converging uniformly must converge to a continuous function; a lemma giving sufficient conditions for uniform convergence.

Lecture 36: Definitions of *disk of convergence, circle of convergence, radius of convergence*; discussion of the type of sets on which a series can converge; proofs that inside the disk of convergence a series converges absolutely and uniformly to a continuous function.

Lecture 37: Proof that power series converge to analytic functions within their disks of convergence.

Lecture 38: Proof that power series representations are unique; discussion of multiplication and division of power series.

Lecture 39: Definition and examples of *isolated singularity*; definition, discussion and computation of *residues*.

Lecture 40: Proof of Cauchy's Residue Theorem, relating the integral of a function along a contour with the sum of the residues inside the contour; proof of a theorem relating the integral of a function along a contour with the residue at 0 of an associated function.

Lecture 41: Discussion of the three types of isolated singular points; definitions of *principal part*, *removable singularity*, *pole of order* m, *simple pole*, *essential singularity*; discussion of the behavior of a function near a pole of order m; proof that near a pole, z_0, of order m, the function f can be written as $\phi(z)(z-z_0)^{-m}$ where ϕ is analytic and non-zero at z_0.

Lecture 42: Discussion of the behavior of functions near removable singularities; proof that at a removable singularity the function can be defined so that it's analytic; proof that boundedness in a neighborhood is equivalent to singularity being removable.

Lecture 43: Discussion of the behavior of functions near essential singularities; proof of the Casorati-Weierstrass theorem (that in every neighborhood of an essential singularity, a function comes arbitrarily close to every complex number); statement of Picard's theorem (that in every neighborhood of an essential singularity a function takes on every finite value, with one possible exception, infinitely many times); proof that in every neighborhood of 0, $e^{1/z}$ takes on all values except zero.

Lecture 44: Definition of a *zero of order* m; proof of lemma characterizing functions with a zero order m; proof of theorem relating zeros of $q(z)$ to poles of $1/q(z)$; calculation of residues at simple poles for functions of the form $p(z)/q(z)$.

Lecture 45: Proof that a function which is analytic on a domain is uniquely determined by its values on any subset with an accumulation point in that domain.